Cubic Spline Numerical Solution of an Ablation Problem with Convective Backface Cooling

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Introduction

THE principal characteristics of high-speed atmospheric re-entry are high aerodynamic heating and limited flight duration. The process of absorbing the heat created by aerodynamic heating allowing a portion of the solid surface to melt or sublimate is known as ablation. In view of the fact that the duration of re-entry is relatively short, the ablation problem has been usually treated as a problem with a semi-infinite region (for example, Refs. 1 and 2).

In practice, however, for a flight body having a relatively thin wall, the effect of internal cooling on the ablation may be significant. The present Note deals with this aspect of the problem.

The prime characteristic of the ablation process is the existence of a moving boundary, whose location is unknown, and which has to be determined as a part of the solution. The moving boundary problem is nonlinear and awkward in mathematical treatment. Exact solutions to such problems are presently restricted to only a few idealized solutions.[3] In order to simplify the problem, the classical ablation model, the melting of a solid at a fixed wall temperature with complete removal of melt and with constant material properties, is used. More modern models of ablation for re-entry problems exist, but are not covered in the present study.

The ablation process for studying re-entry problems in the present Note may be applicable to nuclear reactor vessel accidents involving melting on one side of a wall and convective cooling on the other side.

Mathematical Formulation of the Ablation Problem

We consider a large solid plate of thickness L, initially at a uniform temperature T_0, which is lower than the melting point T_s of the plate material. At time $t=0$, one surface of the plate, designated as $x=0$, is subjected to an external heat flux $q_0(t)$, while the opposite surface at $x=L$ is cooled convectively by a medium having a uniform temperature T_∞. The heat losses from the plate edges are assumed to be relatively small in comparison with the heat transferred by convection from the surface at $x=L$, so that the heat transfer process may be treated as being unidimensional. The ablation process occurs when the temperature at the surface $x=0$ reaches its melting point. For the purpose of analysis, it is assumed that during the ablation process the molten material is removed instantaneously and completely upon its formation so that the moving melting surface acts as a new boundary upon which the external heat flux $q_0(t)$ is impressed. To simplify the problem, it is further assumed that the material properties may be considered as being constant.

The nondimensional system of equations formulating the ablation problem may be described as follows:

1) Preablation process

$$\frac{\partial\Theta}{\partial\tau}=\frac{\partial^2\Theta}{\partial y^2},\quad 0<\tau<\tau_p,\quad 0<y<1 \tag{1}$$

$$\Theta(y,0)=\Theta_0 \tag{2}$$

$$\frac{\partial\Theta(0,\tau)}{\partial y}=-Q(\tau) \tag{3}$$

$$\frac{\partial\Theta(1,\tau)}{\partial y}=-Bi[\Theta(1,\tau)+1] \tag{4}$$

2) Ablation process

$$\frac{\partial\Theta}{\partial\tau}=\frac{1-y}{1-S(\tau)}\frac{\mathrm{d}S(\tau)}{\mathrm{d}\tau}\frac{\partial\Theta}{\partial y}+\frac{1}{[1-S(\tau)]^2}\frac{\partial^2\Theta}{\partial y^2}$$

$$\tau_p<\tau<\infty,\quad 0<y<1 \tag{5}$$

$$\Theta(y,\tau_p^+)=\Theta(y,\tau_p^-) \tag{6}$$

$$\Theta(0,\tau)=0 \tag{7}$$

$$\frac{\partial\Theta(1,\tau)}{\partial y}=-Bi[1-S(\tau)][\Theta(1,\tau)+1] \tag{8}$$

and

$$\frac{\mathrm{d}S(\tau)}{\mathrm{d}\tau}=St\left[Q(\tau)+\frac{1}{1-S(\tau)}\frac{\partial\Theta(S,\tau)}{\partial y}\right] \tag{9}$$

The dimensionless variables and parameters in Eqs. (1-9) are defined as

$$\Theta=-(T-T_s)/(T_\infty-T_s) \tag{10}$$

$$\Theta_0=-(T_0-T_s)/(T_\infty-T_s) \tag{11}$$

$$y=[x-s(t)]/[L-s(t)] \tag{12}$$

$$\tau=at/L^2 \tag{13}$$

$$\tau_p=at_p/L^2 \tag{14}$$

$$S(\tau)=s(t)/L \tag{15}$$

$$Q(\tau)=q_0(t)L/[k(T_s-T_\infty)] \tag{16}$$

$$Bi=hL/k \tag{17}$$

$$St=[k(T_s-T_\infty)]/a\rho L^* \tag{18}$$

Where $s(t)$ is the location of the ablation front, a the thermal diffusivity, t_p the time at which the temperature at $x=0$ reaches the melting point T_s, k the thermal conductivity, h the heat transfer coefficient, ρ the density, and L^* the specific latent heat of fusion.

Cubic Spline Numerical Solution

The cubic spline integration technique is a numerical method which has been applied to solve problems in fluid mechanics by, for example, Rubin and Graves,[4] Panton and Sallee,[5] and Wang and Kahawita.[6,7] The principal advantages of using a cubic spline procedure are as follows:

1) The requirement of a uniform mesh is not necessary. However, for a uniform mesh, the spline approximation is

Received May 15, 1983; revision received Oct. 4, 1983.

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fourth-order accurate for the first derivative while being third-order for a nonuniform grid. The second derivative is approximated to second order for uniform as well as nonuniform grids.

2) Since the value of first or second derivatives may be evaluated directly, boundary conditions containing derivatives may be directly incorporated into the solution procedure.

3) The governing matrix system obtained with the implicit formulation is always tridiagonal, thus facilitating the inversion procedure.

4) The matrix system obtained may be reduced to a scalar set of equations that contain values of either the function itself, its first derivative, or its second derivative at the node points while maintaining a tridiagonal formulation.

For solving the system of equations, an implicit form of the cubic spline integration technique[6,7] is used. In discretized form, Eqs. (1-9) may be expressed as follows:

$$\frac{\Theta_i^{n+1}-\Theta_i^n}{\Delta\tau}=M_i^{n+1}, \qquad i=1,\ N+1 \tag{19}$$

$$\Theta_i^0=\Theta_0, \qquad i=1,\ N+1 \tag{20}$$

$$m_i^{n+1}=-Q^{n+1} \tag{21}$$

$$m_{N+1}^{n+1}=-Bi(\Theta_{N+1}^{n+1}+1) \tag{22}$$

$$\frac{\Theta_i^{n+1}-\Theta_i^n}{\Delta\tau}=\frac{1-y_i}{1-S^n}\left(\frac{dS}{d\tau}\right)^n m_i^{n+1}+\frac{1}{(1-S^n)^2}M_i^{n+1}, \qquad i=1,\ N+1 \tag{23}$$

$$\Theta_i(\tau_p^+)=\Theta_i(\tau_p^-) \tag{24}$$

$$\Theta_1^{n+1}=0 \tag{25}$$

$$m_{N+1}^{n+1}=-Bi(1-S^n)(\Theta_{N+1}^{n+1}+1) \tag{26}$$

$$\left(\frac{dS}{d\tau}\right)^{n+1}=St\left(Q^{n+1}+\frac{1}{1-S^n}m_1^{n+1}\right) \tag{27}$$

where

$$m=\frac{\partial\Theta}{\partial y} \tag{28}$$

and

$$M=\frac{\partial^2\Theta}{\partial y^2} \tag{29}$$

The subscript i refers to the ith grid point along the y direction with $i=1,\ N+1$. The superscript n similarly refers to the nth time step. Equations (19-22) were solved by using a tridiagonal system that contains only the first derivatives m_i^{n+1}. However, Eqs. (23-26) were solved by utilizing a tridiagonal system containing only the function Θ_i^{n+1} (see Appendix of Ref. 6). For the determination of S, the following Taylor series expansion, which includes the term containing the second derivative, was used:

$$S^{n+1}=S^n+\Delta\tau\left(\frac{dS}{d\tau}\right)^{n+1}\left[1+\frac{\Delta\tau St}{2}\frac{m_1^{n+1}}{(1-S^n)^2}\right]+\frac{(\Delta\tau)^2 St}{2}\left(\frac{dQ}{d\tau}\right)^{n+1} \tag{30}$$

For the numerical integration, a nonuniform mesh with six grid points was used. The numerical error involved in the coarsely spaced grid is about 1.0%.

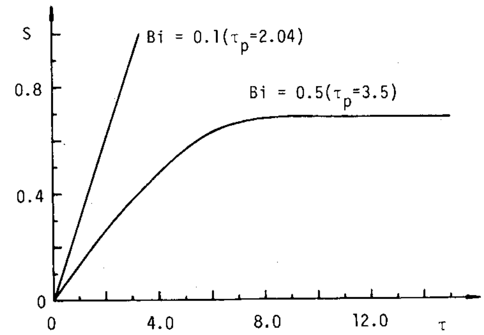

Fig. 1 Dimensionless moving boundary S as a function of dimensionless time τ with Bi as a parameter for $\Theta_0=-1.0$, $Q=0.5$, and $St=1.0$.

To illustrate the effect of the convective heat transfer on the location of the moving ablation boundary, Fig. 1 indicates the dimensionless moving ablation boundary S as a function of the dimensionless time τ with Biot number Bi as a parameter for $\Theta_0=-1.0$, $Q=0.5$, and $St=1.0$. The parameter Bi represents the ratio of the conductive thermal resistance in the wall to the convective thermal resistance at the surface. The smaller the value of the convective resistance, or the higher the value of the parameter Bi, the more efficient is the heat transfer process at the convectively cooled wall, resulting in a smaller penetration distance of the ablation boundary as shown in Fig. 1. For the case of $Bi=0.5$, the maximum penetration of the ablation boundary is limited to about $S_{max}=0.65$; at this position, the external heat flux is balanced by the internal convective cooling. For the case of $Bi=1.0$, the internal convective heat transfer is so high that the temperature at the outer surface of the wall never reaches the melting point. There is, therefore, no ablation. In Fig. 1, τ_p represents the dimensionless preablation time. It is expected that the higher the value of the parameter Bi, or the more efficient the convective heat transfer process, the longer the preablation time.

Conclusions

The system of equations describing the problem of ablation on a thin wall with convective cooling has been integrated numerically using a cubic spline technique. Based on the fact that a nonuniform mesh with six grid points provides a numerical error of about 1.0%, the method has been found to be computationally efficient.

Numerical results indicate that convective cooling is an important factor in reducing the thickness of ablation.

Acknowledgment

The present work is supported by the National Sciences and Engineering Research Council of Canada under Grants A7929 and A8846.

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